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Dynamic models for gas and water flow in pipelines

Dynamiczne modele przepływu gazu i wody w rurociągach

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Abstract
Mathematical models for transient flow of gas and water are described. The basic equations describing the transient flow of fluid in pipe are derived from an equation of motion, an equation of continuity and state equation. For water, mathematical models have been formulated based on compressible and incompressible flow theory. The simplified models for gas were obtained by neglecting some terms in the basic equations..

Słowa kluczowe: *modele matematyczne, przepływ gazu, przepływ wody, przepływ nieustalony, przepływ w rurociagach*

Streszczenie
Omówiono modele matematyczne nieustalonego przepływu wody i gazu. Równania ogólne opisujące nieustalony przepływ płynu w rurze wyprowadzono na podstawie równania ciągłości, zachowania momentu oraz równania stanu. W przypadku wody modele sformułowano zarówno dla przypadku ścisliwego jak rownież niescisliwego. Uproszczone modele opisujace nieustalony przepływ gazu otrzymano pomijając niektóre człony w równaniach wyjsciowych.

Introduction

The basic equations describing the transient flow of fluid in pipes are derived from an equation of motion (or momentum),an equation of continuity and state equation. In practice the form of the mathematical models varies with the assumptions made as regards the conditions of operation of the fluid pipes or networks. In the case of mathematical models for transient simulation of water networks it has to be decided whether the model should be based on compressible (water hammer) or incompressible (rigid column) flow theory. An incompressible models of flow are described by ordinary differential equations. Compressible models for gas and water are described by partial differential equations or a system of such equations. On the models, two contradictory constraints are imposed. It is required that on the one hand the description of the phenomenon be accurate, and on the other that it be as simple as possible so that the computational means necessary for solving this model be reasonable. As a rule simplified models are sought which present a reasonable compromise between the accuracy of description and the costs of solution. The simplified models are obtained by neglecting some terms in the basic model as a result of a quantitive estimation of the particular elements of the equation for some given conditions of operation of the network.

Depending on the degree of simplification with respect to the set of basic equations, the equations may be linear or quite generally nonlinear. They may be parabolic or hyperbolic of the 1st or 2nd order.

2. Conservation of mass: continuity equation

The law of conservation of mass simply states that mass may neither be created nor destroyed. Thus, the mass of the system remains constant.

Generally, the continuity equation is expressed (Osiadacz, 1987) in the form:

$$\frac{\partial w}{\partial x} + \frac{1}{c^2\rho}\frac{\partial p}{\partial t} = 0 \qquad (1)$$

where:
w – is the flow velocity,
p – is the pressure of fluid,
c – is the sonic velocity,
ρ – is the density

Substituting Q = w A, we have:

$$\frac{A}{c^2\rho}\frac{\partial p}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (2)$$

or

$$\frac{A}{c^2}\frac{\partial p}{\partial t} + \frac{\partial M}{\partial x} = 0 \qquad (3)$$

where: A – is the cross-section area of the pipe,
M – is the mass flow of fluid.
Noting that

$$p = \rho g H$$

where:
H – is the piezometric head at the centerline of the pipe above the specified datum,
g – is the acceleration due to gravity,
eq. (2) takes the form:

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$$\frac{c^2}{gA}\frac{\partial Q}{\partial x}+\frac{\partial H}{\partial t}=0 \tag{4}$$

2.1 Sonic velocity in fluids

The sonic velocity c the velocity of propagation of a pressure wave in a fluid, is expressed by many different formulas (Sharp (1981), Stephenson (1984), Watters (1979), Wylie, Streeter (1976). In (Halliwell, 1963), the following general expression for c is presented:

$$c^2 = \frac{K}{\rho\left[1+(K/E)\ \psi\right]} \tag{5}$$

where:
ψ – is a nondimensional parameter that depends upon the elastic properties of the pipe:
– (for rigid pipe ψ = 0),
K – is the bulk modulus of elasticity of a fluid,
E – is the Young's modulus of elasticity.
Since for rigid pipe ψ = 0, then,

$$c^2 = K/\rho \tag{6}$$

Taking into account the relation:

$$K = \frac{\Delta p}{\Delta\rho/\rho} \tag{7}$$

we obtain

$$c^2 = \frac{dp}{d\rho} \tag{8}$$

Equation (8) means that the sonic velocity is related to the compressibility of the fluid (dρ/dp). The density of the fluid is given by

$$\rho = \frac{1}{v} \tag{9}$$

where:
v – volume per unit mass.
Differentiate eq. (9) to give

$$\frac{d\rho}{dv} = -\frac{1}{v^2} \tag{10}$$

and rewrite in the form

$$\frac{dp}{d\rho} = -v^2\frac{dp}{dv} \tag{11}$$

When ideal gases are compressed or expanded they obey the following equation

$$pv^k = \text{const.} \tag{12}$$

In equation (12), k = 1 for an isothermal change of state and $k = \kappa = c_p/c_v$.
where:
c_p – is the heat capacity at constant pressure,
c_v – is the heat capacity at constant volume,
κ – is the isentropic exponent.

Differentiate eq. (10) to give:

$$\frac{dp}{dv} = -k\frac{E}{v} \tag{13}$$

Combine eqs. (11) and (12) to give

$$\frac{dp}{d\rho} = k\ p\ v \tag{14}$$

putting eq.(14) into eq. (8) we have

$$c^2 = k\ p\ v = k\ p/\rho \tag{15}$$

For isothermal conditions, eq. (15) becomes

$$c^2 = p/\rho \tag{16}$$

For adiabatic conditions, eq. (15) becomes

$$c^2 = \kappa\ p/\rho \tag{17}$$

3. Newton’s second law of motion: momentum equation

The known form of Newton's second law is that the force acting on a fluid particle or system of particles of fixed mass at a certain instant is equal to the rate of change of momentum of the particle (system of particles) at that instant. Consider only the streamline direction,

$$\sum F_x = \frac{d}{dt}(m w) \tag{18}$$

where:
F_x – is the component of the forces acting on the element of fluid in the direction of motion.
According to (Osiadacz, 1987)

$$\frac{d}{dt}(m\ w) = \frac{\partial}{\partial t}(\rho\ A\ w\ dx) + \frac{\partial}{\partial x}(\rho\ A\ w^2\ dx) \tag{19}$$

The net force (in the direction of flow) consists of the algebraic sum of the x component of all the forces that act on the fluid within the control surface.

The forces are the following:
– pressure force:

$$p\ A - \left(p + \frac{\partial p}{\partial x}dx\right)A = -\frac{\partial p}{\partial x}A\ dx \tag{20}$$

– shear force which is due to the friction

$$\text{shear force} = -\rho\ A w^2\ 2f\ \frac{dx}{D} \tag{21}$$

– the component of the net body force acting on the gas within the control volume is

$$F\ \rho\ A\ dx\ \sin\alpha \tag{22}$$

where:
f – is the Fening’s friction factor,
ρ – is the engle between the horizontal and direction x,
D – is the diameter of the pipe.

Substituting eqs. (19), (20), (21) and (22) into eq. (18) we have

$$-\frac{\partial p}{\partial x} - \rho w^2\frac{2f}{D} - g\ \rho\sin\alpha = \frac{\partial}{\partial t}(\rho\ w) + \frac{\partial}{\partial x}(\rho\ w^2) \tag{23}$$

Dividing the above equation by ρ, we obtain

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} - w^2\frac{2f}{D} - g\sin\alpha = \frac{\partial w}{\partial t} + \frac{\partial w^2}{\partial x} \quad (24)$$

Noting that H = p/(ρg), eq. (24) becomes

$$g\frac{\partial H}{\partial x} - w^2\frac{2f}{D} - g\sin\alpha = \frac{\partial w}{\partial t} + \frac{\partial w^2}{\partial x} \quad (25)$$

or

$$gA\frac{\partial H}{\partial x} - \frac{Q|Q|}{A}\frac{2f}{D} - Ag\sin\alpha = \frac{\partial Q}{\partial t} + w\frac{\partial Q}{\partial x} \quad (26)$$

In most transient problems for water the term $\partial(\rho w^2)/\partial x$ is much smaller then the term $\partial(\rho w)/\partial t$.

Therefore, the former term may be neglected.

Equation (23) can be written as:

$$-\frac{\partial p}{\partial x} - \rho w^2\frac{2f}{D} - \rho g\sin\alpha = \frac{\partial}{\partial t}(\rho w) \quad (27)$$

4. Mathematical models for gas

The following assumptions are made in developing the equations for transient gas flow in pipeline:

- flow is isothermal,
- expansion at pipe wall may be neglected,
- one dimensional flow relations are used,
- the gas compressibility is assumed constant over the range of a single problem,
- the cross sectional area change slowly along the path of stream of gas,
- the radius of curvature of the pipe is large in comparison to diameter,
- the shapes of velocity and temperature profile are approximately constant along the pipe,
- for one dimensional flow of gas, pressure, density, velocity and etc, are only functions of time and the distance along the axis of the pipe.

Generally, the transient flow of gas in a pipe is described by the system of equations (4) and (23), i.e.

$$\frac{A}{c^2}\frac{\partial p}{\partial t} + \frac{\partial M}{\partial x} = 0$$

where: $c = \sqrt{p/\rho}$

and

$$\frac{\partial p}{\partial x} + \rho w^2\frac{2f}{D} + g\rho\sin\alpha + \frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho w^2) = 0$$

respectively.

We impose on the above equations two contradictory constraints. It is required that on the one part the description on the phenomenon be accurate, and on the other that it be possibly simple so that the computational means necessary for solving this model be reasonable. As a rule simplified models are sought which present a reasonable compromise between the accuracy of description and the costs of solution. The simplified models are obtained by neglecting some terms in the basic (accurate) equation as a result of a quantitive estimation of the particular elements of the equation for some given conditions of operation of the pipeline. This means that the model of transient flow used for similation should be fitted to the given conditions of operation of the pipe. A necessary condition for proper selection of the model is therefore an earlier analysis of these conditions. Estimation of the particular terms of the equation (22) for given operating conditions and a given geometry of the pipe is given in (Osiadacz 1987). The character of the results cannot be general. It can only be the starting point which allows the formulation of the hypothesis that in the case when boundary conditions do not change rapidly, or capacity of the pipe is relatively large, transient flow through the horizontal pipe can be represented by a set of the following equations:

$$\begin{cases} \frac{A}{c^2}\frac{\partial p}{\partial t} + \frac{\partial M}{\partial x} = 0 \\ \frac{\partial p}{\partial x} + \frac{2f\rho w^2}{D} = 0 \end{cases} \quad (53)$$

Taking into account that

$$M = \rho w A = \rho Q = \rho_s Q_s$$

(where: the subscript n refers to quantities at standard conditions of pressure p_s0.1MPa and temperature T_s=273K)
and $p = c^2\rho$, the above system of equations takes the form:

$$\begin{cases} \frac{\partial p}{\partial t} = -\frac{c^2\rho_s}{A}\frac{\partial Q_s}{\partial x} \\ \frac{\partial p}{\partial x} = -\frac{2f\rho_s^2c^2Q_s^2}{DA^2p} \end{cases} \quad (54)$$

In turn the system of equations (54) can be transformed into:

$$\begin{cases} \frac{\partial p}{\partial t} = -\frac{c^2\rho_s}{A}\frac{\partial Q_s}{\partial x} \\ \frac{\partial p^2}{\partial x} = -\frac{4f\rho_s^2c^2}{DA^2}|Q_s|Q_s \end{cases} \quad (55)$$

Writing the second equation of eqs. (54) in the form

$$\frac{\partial p^2}{\partial x} = -\frac{4f\rho_s^2c^2Q_s^2}{DA^2}$$

and differentiating with respect to x we get:

$$\frac{\partial^2 p^2}{\partial x^2} = -\frac{4f\rho_s^2c^2}{DA^2}2Q_s\frac{\partial Q_s}{\partial x}$$

Taking account of the first equation in the system (54) we get:

$$\frac{\partial^2 p^2}{\partial x^2} = \frac{4f\rho_s^2c^2}{DA^2}2Q_s\frac{A}{c^2\rho_s}\frac{\partial p}{\partial t}$$

Finally, the biquadratic model is obtained

$$\frac{\partial^2 p^2}{\partial x^2} = a\frac{\partial p^2}{\partial t} \quad (56)$$

where:

$$\alpha = \frac{4fQ}{DAc^2}$$

This is a nonlinear parabolic model. If we assume that α = const, (this is true for the case when variations of flow through pipeline are small) we will get a linear equation with respect to p^2. Assuming that Q(x,t) is averaged over length in every time interval Δt, we get a second order parabolic partial differential equation linear with respect to p^2 in every time step. Using eqs. (54), we can get a linear model with respect to p in the following way.

Equation

$$\frac{\partial p}{\partial x} = -\frac{2\ f\ \rho_s^2\ c^2\ Q_s^2}{D\ A^2\ p}$$

can be written as

$$\frac{\partial p}{\partial x} + \lambda_1\ Q_s = 0 \qquad (57)$$

where:

$$\lambda_1 = \lambda_1(p, Q_s) = \frac{2\ f\ \rho_s^2\ c^2\ |Q_s|}{D\ A^2\ p}$$

We can trasform eq. (57) to the following form:

$$Q_s = -\frac{1}{\lambda_1}\frac{\partial p}{\partial x} \qquad (58)$$

Substituting eq. (58) into the first of eqs. (54) we get: Assuming that $\lambda_1(x,t)$ is averaged over length in every time step Δt, we get a second order parabolic partial differential equation linear with respect to p in every time step. If we consider the relation

$$M = Q_s\ \rho_s$$

we have

$$\frac{\partial p}{\partial x} + \lambda_2\ M = 0 \qquad (59)$$

where:

$$\lambda_2 = \lambda_2(p, M) = \frac{2\ f\ c^2\ |M|}{D\ A^2\ p}$$

$$M = -\frac{1}{\lambda}\ \frac{\partial p}{\partial x} \qquad (60)$$

and finally

$$\frac{\partial p}{\partial t} = \frac{c^2}{A\ \lambda}\ \frac{\partial^2 p}{\partial x^2} \qquad (61)$$

Assuming that

$$\frac{2 f \rho w^2}{D} \approx \left[\frac{2 f w}{D}\right]_{ave} \rho\ w = \beta\ \rho\ w \quad \text{with} \quad \beta = \left[\frac{2 f w}{D}\right]_{ave}$$

the equation

$$\frac{\partial p}{\partial x} + \frac{2 f \rho w^2}{D} = 0$$

is linearized (Osiadacz, 1987).

After linearization, the system of equations (53) takes the form:

$$\begin{cases} \dfrac{A}{c^2}\ \dfrac{\partial p}{\partial t} + \dfrac{\partial M}{\partial x} = 0 \\ \dfrac{\partial p}{\partial x} + \beta\ \rho\ w = 0 \end{cases} \qquad (62)$$

where:

$$\beta = \frac{2\ f}{D}\ \overline{w}$$

$$\overline{w} = \frac{2}{3}\ \frac{w_2^2 + w_1\ w_2 - 2\ w_1^2}{w_2 - w_1}$$

It was assumed that the gas velocity in a pipe varies from w_1 to w_2. After transformation, we have the linear equation of diffusion:

$$\frac{\partial p}{\partial t} = \frac{c^2}{\beta}\ \frac{\partial^2 p}{\partial x^2} \qquad (63)$$

If the changes of the gas pipeline load are rapid, it becomes necessary to provide additionally in the equation of gas motion for the term characterizing the inertia of the flowing gas. We should then consider the system of equations in the form:

$$\begin{cases} \dfrac{A}{c^2}\ \dfrac{\partial p}{\partial t} + \dfrac{\partial M}{\partial x} = 0 \\ \dfrac{\partial p}{\partial x} + \dfrac{\partial(\rho w)}{\partial t} + \dfrac{2\ f\ \rho\ w^2}{D} = 0 \end{cases} \qquad (64)$$

After similar linearization, second equation of (64) we can write in the following way:

$$\frac{\partial p}{\partial x} + \frac{\partial(\rho w)}{\partial t} + \beta\ \rho\ w = 0 \qquad (65)$$

Next, differentiating eq.(65) with respect to t and first equation of (64) with respect to x, we obtain respectively:

$$\frac{\partial^2 p}{\partial x\ \partial t} + \frac{\partial^2(\rho w)}{\partial t^2} + \beta\ \frac{\partial(\rho w)}{\partial t} = 0 \qquad (66)$$

and

$$\frac{A}{c^2}\ \frac{\partial^2 p}{\partial t\ \partial x} + \frac{\partial^2 M}{\partial x^2} = 0 \qquad (67)$$

Finally, taking into account eqs,(66) and (67), we have:

$$c^2\ \frac{\partial^2 M}{\partial x^2} - \frac{\partial^2 M}{\partial t^2} - \beta\ \frac{\partial M}{\partial t} = 0 \qquad (68)$$

– second order hyperbolic equation.

In the case of short gas pipelines or when friction is very small we can neglect the term $(2\ f\ \rho\ w^2)\ /\ D$ if rapid load changes occur preserving the term $\partial(\rho w)/\partial t$, since the damping of rapid changes of flow takes place only if friction forces occur. In thissituation we get the following system of equations:

$$\begin{cases} \dfrac{A}{c^2}\ \dfrac{\partial p}{\partial t} = -\dfrac{\partial M}{\partial x} \\ \dfrac{\partial p}{\partial x} = -\dfrac{\partial(\rho w)}{\partial t} \end{cases} \qquad (69)$$

After transformations we get:

$$\frac{\partial^2 p}{\partial t^2} = c^2\ \frac{\partial^2 p}{\partial x^2}$$

i.e. a second order hyperbolic linear equation known as a wave equation. In the simplified models discussed above it was assumed that the gas pipeline is horizontal ($\rho g \sin\alpha = 0$).

This points to significant simplifications of the equations with respect to the basic set of equations. However, there are publications in which authors use more complex models, in which term $\rho g \sin\alpha$ is not neglected.

5. Mathematical models for water

Prior to developing a mathematical model for the simulation of transients, it has to be decided whether the model should be based on compressible or incompressible flow analysis. The behaviour of water hammer pressure-transients is governed partly by the inertia of the moving water and partly by the combined elasticity of the pipework and fluid, i.e. the water plus any free air that may be present. Compressible-flow theory corresponds to the case in which both, inertia and elasticity must be considered. Incompressible flow theory (rigid column theory) relates to neglecting the elastic effect. Incompressible flow theory gives reasonable results provided that:

- (i) the timescale t for the important forcing conditions to change (e.g. a valve to shut or a pump to run down) is long compared to the time-scale t for a pressure-wave to treverse the system,
- (ii) these forcing conditions change smoothly.

Incompressible flow theory is mathematically simpler than compressible flow theory, since ordinary rather then partial differential equations are handled within pipe segments.

Longer timesteps are possible, since the Courant condition $c\,\Delta t \leq \Delta x$ is no longer an obstacle: nor to the corresponding technical computational problems of handling short elements arise. In general, a compressible flow theory calculation can be regarded as being made up of the corresponding incompressible flow theory result plus a superposed oscillation coming from the elastic effects. This oscillatory part depends on the sound speed used in the calculation: this may be easy to predict since it depends strongly on the amount of free air if any present in the water. Using incompressible flow theory corresponds to calculating the underlying trend only.

5.1 Rigid water column theory

Rigid water column theory corresponds to the case in which inertia of the moving water and elasticity of the pipework and fluid are neglected. Below, two simple examples are given in each of which flow is described by an ordinary differential equation.

Example 1

In Fig. 1 a simple system consisting of reservoir, pipe and valve is shown. If the valve is closed, the pressure in the pipe is equal to piezometric head H_0. If the valve is opened suddenly, the pressure at the valve drops instantly to zero, and the fluid begins to accelerate. By integrating equation (27) between x = 0 and x = L, (where L is the length of the pipe) and multiplying by dx we get

$$-\frac{1}{\rho}\int_0^L \frac{\partial p}{\partial x}\,dx - \int_0^L \frac{2\,f\,w^2}{D}\,dx - \int_0^L g\sin\alpha\,dx = \int_0^L \frac{\partial w}{\partial t}\,dx \qquad (28)$$

For a horizontal constant-diameter pipe, $\sin\alpha = 0$, and w is a function of time only.

The result is

$$\frac{p_{x=0} - p_{x=L}}{\rho} - \frac{2\,f\,L\,w^2}{D} = L\,\frac{dw}{dt} \qquad (29)$$

or

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} - \frac{2\,f\,L\,w^2}{g\,D} = \frac{L}{g}\,\frac{dw}{dt} \qquad (30)$$

where: γ – is a specific weight

This equation is reduced to the following form:

$$H_0 - \frac{2\,f\,L\,w^2}{g\,D} = \frac{L}{g}\,\frac{dw}{dt} \qquad (31)$$

The integration of eq. (31) (because the pressure head p_1/γ=const = H_0 and because p_1/γ=0, for t>0) determines the time necessary to accelerate the flow to a given velocity, w;

$$t = \frac{2\,L\,w_0}{g\,H_0}\,\log\frac{w_0 + w}{w_0 - w} \qquad (32)$$

where:

$$w = \sqrt{g\,H_0\,D}\,/\left(2\,f\,L\right)$$

Example 2

If the valve is open, the pressure upstream of the valve is determined by loss characteristics of the flow through the valve. This causes difficulties, in the case of rapid valve closure. At time t = 0, the velocity is w_0 (the steady-state velocity). The equation characterizing this problem is the following:

$$H_0 - \frac{p_2}{\gamma} - \frac{2\,f\,L\,w^2}{g\,D} = \frac{L}{g}\,\frac{dw}{dt} \qquad (33)$$

Since we have two dependent variables; p_2 and w, we need another equation which is derived from an energy equation across the valve:

$$\frac{p_2}{\gamma} = K_L\,\frac{w^2}{2\,g} \qquad (34)$$

where:
K_L – is the valve loss coefficient

4.1.1 Basic equations for unsteady flow in series pipes

We can write equation (30) in the form:

$$\left(H_1 - H_2\right) - \frac{2\,f\,L\,w^2}{g\,D} = \frac{L}{g}\,\frac{dw}{dt} \qquad (35)$$

This equation can be rewritten in terms of flow Q as

$$S\,\frac{dQ}{dt} = g\,h - R\,Q\,|Q| \qquad (36)$$

where:

$$Q = w\,A,\quad h = H_1 - H_2,\quad S = L/A,\quad R = \frac{2\,f\,L}{D\,A^2}$$

R and S are called the resistance and inertia respectively.
For the i-th pipe we have:

$$S_i\,\frac{dQ_i}{dt} = g\,h_i - R_i\,|Q_i|\,Q_i \qquad (37)$$

If instead of a single pipe we have a chain of n pipes in series joining nodal points 1,2,...,m+1 at which the heads are $H_1, H_2, \ldots, H_{m+1}$, then for each pipe

$$S_i\,\frac{dQ}{dt} = g\,h_i - R_i\,|Q|\,Q \qquad h_i = H_i - H_{i+1} \qquad (38)$$

Adding these equations, it will be seen that the chain behaves like a single element of total resistance R_{tot}, and inertia S_{tot} under a head difference h_{tot}, where

$$R_{tot} = \sum_{i=1}^{m} R_i \qquad S_{tot} = \sum_{i=1}^{m} S_i \qquad h_{tot} = \sum_{i=1}^{m} h_i$$

4.1.2 Basic equations for unsteady flow in parallel pipes

In the development of an equivalent pipe for a parallel pipe system we will consider the case shown in Fig. 2.

We assume that:

$$R_1\,Q_1^2 \;=\; R_2\,Q_2^2 \;=\; R_3\,Q_3^2 \;=\; R_{eq}\,Q_{eq}^2 \qquad (39)$$

where: $Q_1 + Q_2 + Q_3 = Q_{eq}$

Substituting eq. (39) into the above expression for continuity gives:

$$\frac{R_{eq}}{R_1}Q_{eq} + \frac{R_{eq}}{R_2}Q_{eq} + \frac{R_{eq}}{R_3}Q_{eq} = Q_{eq} \qquad (40)$$

Dividing out Q_{eq} and regrouping

$$\frac{1}{\sqrt{R_1}} + \frac{1}{\sqrt{R_2}} + \frac{1}{\sqrt{R_3}} = \frac{1}{\sqrt{R_{eq}}} \qquad \frac{1}{\sqrt{R_{eq}}} = \sum_{i=1}^{N}\frac{1}{\sqrt{R_i}} \qquad (41)$$

Now, we write a dynamic equation for each pipe:

$$g\,h_1 - R_1\,Q_1\left|Q_1\right| = S_1\frac{dQ_1}{dt}$$

$$g\,h_2 - R_2\,Q_2\left|Q_2\right| = S_2\frac{dQ_2}{dt}$$

$$g\,h_3 - R_3\,Q_3\left|Q_3\right| = S_3\frac{dQ_3}{dt}$$

$$g\,h_{eq} - R_{eq}\,Q_{eq}\left|Q_{eq}\right| = S_{eq}\frac{dQ_{eq}}{dt}$$

Because of friction loss equivalence, the left hand side of the above equations are equal, giving:

$$S_1\frac{dQ_1}{dt} + S_2\frac{dQ_2}{dt} + S_3\frac{dQ_3}{dt} = S_{eq}\frac{dQ_{eq}}{dt} \qquad (42)$$

Writing the equation of continuity in differential form:

$$dQ_1 + dQ_2 + dQ_3 = dQ_{eq} \qquad (43)$$

and substituting expressions for the dynamic equation gives:

$$\frac{S_{eq}}{S_1}dQ_{eq} + \frac{S_{eq}}{S_2}dQ_{eq} + \frac{S_{eq}}{S_3}dQ_{eq} = dQ_{eq} \qquad (44)$$

Dividing out dQ_{eq},

$$\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} = \frac{1}{S_{eq}} \quad \text{or} \quad \frac{1}{S_{eq}} = \sum_{i=1}^{N}\frac{1}{S_i} \qquad (45)$$

4.2 Water hammer theory

The fundamental equations describing the phenomenon of water hammer are obtained from consideration of mass and of momentum, i.e. eqs. (4) and (26)

$$\begin{cases} \dfrac{c^2}{g\,A}\dfrac{\partial Q}{\partial x} + \dfrac{\partial H}{\partial t} = 0 \\ g\;A\;\dfrac{\partial H}{\partial x} - \dfrac{Q\left|Q\right|}{A}\dfrac{2\,f}{D} - A\;g\,\sin\alpha = \dfrac{\partial Q}{\partial t} + w\,\dfrac{\partial Q}{\partial x} \end{cases}$$

For horizontal pipes $Ag\sin\alpha = 0$.

Generally, the term $w\,\partial Q/\partial x$ is small compared with $\partial Q/\partial t$ and can be neglected, but it can be accounted for in numerical solutions if necessary, e.g. in flexible plastic piping.

The basic differential water hammer equations including a friction term thus become:

$$\begin{cases} \dfrac{c^2}{g\,A}\dfrac{\partial Q}{\partial x} + \dfrac{\partial H}{\partial t} = 0 \\ g\;A\;\dfrac{\partial H}{\partial x} - \dfrac{Q\left|Q\right|}{2\,A}\dfrac{4\,f}{D} - \dfrac{\partial Q}{\partial t} = 0 \end{cases} \qquad (46)$$

Omitting the friction term the equations become:

$$\begin{cases} \dfrac{c^2}{g\,A}\dfrac{\partial Q}{\partial x} + \dfrac{\partial H}{\partial t} = 0 \\ g\;A\;\dfrac{\partial H}{\partial x} - \dfrac{\partial Q}{\partial t} = 0 \end{cases} \qquad (47)$$

The general solution to these equations is

$$H = H_0 + f_1\left(t - \frac{x}{c}\right) + f_2\left(t + \frac{x}{c}\right) \qquad (48)$$

$$Q = Q_0 + \frac{g\,A}{c}f_1\left(t - \frac{x}{c}\right) + \frac{g\,A}{c}f_2\left(t + \frac{x}{c}\right) \qquad (49)$$

which indicates that pressure and flow changes are propagated at speed +c along the pipe.

4.2.1 Effect of air

The presence of free air in pipelines can reduce the severity of water hammer considerably. Fox (1977) indicates that the speed of an elastic wave with free air is:

$$c = \frac{1}{\sqrt{\rho\left(\dfrac{1}{R} + \dfrac{D}{e\,R} + \dfrac{f_1}{p}\right)}} \qquad (50)$$

where:

f_1 – is the gas fraction by volume.

For large air conten

$$c = \sqrt{g\,H\;/\,f_1} \qquad (51)$$

ts eq. (50) is reduced to the form

The wave speed of a liquid-gas mixture in an elastic pipe may be derived from application of the momentum and continuity equations (Tullis, Streeter and Wylie, 1976) in the following form:

$$c = \frac{\sqrt{K_i\;/\;\rho_{ave}}}{\sqrt{1 + \dfrac{K_\Omega}{E}\dfrac{D}{e}\,C + \left(\text{void fraction}\right)\dfrac{K_\Omega}{K_a}}} \qquad (52)$$

where: subscripts Ω and a refer to properties of liquid and air respectively.

ρ_{ave} – is the average density of the mixture expressed by the relation;

$$\rho_{ave} = \frac{\rho_{\Omega} \ L_{\Omega} + \rho_a \ L_a}{L}$$

where:

L – is the length of the pipe

L_{Ω} – is the length of the pipe containing a liquid

L_a – is the length of the pipe containing an air

C = (5/4 – μ) – for pipe supported at one end only

C = (1 – μ^2) – for both ends fixed,

C = 1 – for a pipe with expansion joints

μ – is the Poisson ratio (for example for steel = 0.3)e.

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